

Wrapping the M Theory Five-Brane on K3

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Abstract

Using a recently constructed M5-brane world-volume action, we demonstrate that wrapping the M5-brane on K3 gives the heterotic string in seven dimensions. To facilitate the comparison, a new version of the world-sheet action for the Narain-compactified heterotic string, with manifest T duality invariance, is formulated.

¹Work supported in part by the U.S. Dept. of Energy under Grant No. DE-FG03-92-ER40701.

1 Introduction

One of the remarkable dualities that was pointed out by Witten two years ago relates heterotic string theory compactified on a torus to seven dimensions and eleven-dimensional M theory (a name that was coined later) compactified on K3 [1]. It was noted that the Narain moduli space of the 7d heterotic theory precisely matches the moduli space of Ricci-flat metrics on K3. Points of enhanced gauge symmetry in the heterotic description correspond to K3's in which two-cycles degenerate. Also, the correspondence of the volume modulus \mathcal{V} of the K3 and the heterotic dilaton ϕ satisfies the scaling law $\mathcal{V} \sim \lambda_H^{4/3}$, where $\lambda_H = \exp \phi$ is the heterotic string coupling constant. Thus, this pair of theories is U dual in the sense that large volume corresponds to strong coupling.

Shortly after this duality was proposed, the matching of various solitons in 7d, viewed from the two dual perspectives, was considered [2, 3].² It was noted, in particular, that the heterotic string in 7d can be understood from the M theory viewpoint as arising from wrapping the five-brane of M theory (or the 'M5-brane') on the K3. A convincing case was made that this give the correct degrees of freedom on the string world sheet. Specifically, the 3 right-moving and 19 left-moving compact chiral bosons on the world sheet arise from the zero modes of the chiral two-form of the M5-brane as a direct consequence of the fact that K3 has 3 self-dual and 19 anti-self-dual two-forms. The purpose of this paper is to study this correspondence in more detail. This turns out to be an interesting, and possibly useful, exercise. In particular, it motivates the formulation of a new world-sheet action for the Narain-compactified heterotic string in which the T duality symmetries are manifest.

Explicit formulas for the M5-brane world-volume action have recently been constructed [4, 5, 6, 7], making the present study possible. Ref. [4] developed a formalism without manifest covariance for describing the chiral two-form gauge field of the M5-brane. A covariant formalism was subsequently applied to this problem by Pasti, Sorokin, and Tonin (PST) [5] using an approach they had developed earlier [8]. This formulation contains an auxiliary scalar field and new gauge invariances.³ The non-covariant formulas of [4] arise in the PST formulation from choosing a particular gauge. This paper will use the covariant PST approach. When four of the dimensions of the M5-brane are wrapped on a K3 space, and

²together with the related 6d problem of matching solitons of type IIA theory on K3 and heterotic theory on T^4 .

³An alternative covariant formulation of the M5-brane field equations, which does not appear to contain this scalar field, has been proposed in ref. [9].

only zero modes of the K3 are retained (as in Kaluza-Klein dimensional reduction), the remaining action describes the two-dimensional world-sheet of a string in seven dimensions. Since the resulting formulas look quite different from anything that can be found in the literature, the first part of this paper is devoted to developing the appropriate description of the heterotic string theory. Then it is relatively straightforward to see that this corresponds to what arises from the double dimensional reduction of the M5-brane on K3. Most of the essential issues already arise for the bosonic degrees of freedom of the M5-brane and the heterotic string. Therefore, in this paper we only consider this truncated problem. It would be desirable, however, to generalize our analysis to include the fermionic degree of freedom so as to incorporate the appropriate target space supersymmetries and world-sheet kappa symmetry.

2 The Heterotic String

There are two main approaches to constructing the world-sheet action of the heterotic string that have been used in the past [10]. In one of them, the internal torus is described in terms of bosonic coordinates. The fact that these bosons are chiral (*i.e.*, the left-movers and right-movers behave differently) is imposed through external constraints. In the second approach these bosonic coordinates are replaced by world-sheet fermions, which are Majorana–Weyl in the 2d sense. What will be most convenient for our purposes is a variant of the first approach. In this variant the coordinates of the Narain torus are still represented by bosonic fields, but the chirality of these fields is achieved through new gauge invariances rather than external constraints.

In order not to confront too many issues at once, the lagrangian of the Narain torus for a flat world-sheet geometry will be described first. Then we will add a world-sheet metric field so as to achieve general coordinate invariance of the world-sheet theory. Finally, the auxiliary world-sheet metric will be eliminated by solving its equations of motion, so as to obtain the appropriate generalization of the Nambu–Goto action.

2.1 Lorentz Invariant Action for the Narain Torus

Consider the Narain compactified heterotic string in a Minkowski space-time with $d = 10 - n$ dimensions. Let these coordinates be denoted by X^m with $m = 0, 1, \dots, d - 1 = 9 - n$. To properly account for all the degrees of freedom, the Narain torus should be described by

$16 + 2n$ bosonic coordinates Y^I , $I = 1, 2, \dots, 16 + 2n$. These will be arranged to describe $26 - d = 16 + n$ left-movers and $10 - d = n$ right-movers. The Y^I are taken to be angular coordinates, with period 2π , so that $Y^I \sim Y^I + 2\pi$, and the conjugate momenta are integers. The actual size and shape of the torus is encoded in a matrix of moduli, denoted M_{IJ} , which will be described below.

The $(16 + 2n)$ -dimensional lattice of allowed momenta should form an even self-dual lattice of signature $(n, 16 + n)$. Let us therefore introduce a matrix

$$\eta = \begin{pmatrix} I_n & 0 \\ 0 & -I_{16+n} \end{pmatrix}, \quad (1)$$

where I_n is the $n \times n$ unit matrix. An even self-dual lattice with this signature has a set of $16 + 2n$ basis vectors V_I , and the symmetric matrix

$$L_{IJ} = V_I^a \eta_{ab} V_J^b \quad (2)$$

characterizes the lattice. A convenient specific choice is

$$L = \Lambda_8 \oplus \Lambda_8 \oplus \sigma \oplus \dots \oplus \sigma, \quad (3)$$

where Λ_8 is the negative of the E_8 Cartan matrix and $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ appears n times. For $n > 0$, an arbitrary even self-dual lattice can be obtained from this one by applying a suitable $O(n, 16 + n)$ transformation. For $n = 0$, this choice is appropriate to the $E_8 \times E_8$ theory. For the $SO(32)$ theory in 10d one would use $L = \Lambda_{16}$, which characterizes the weight lattice of the group $\text{spin}(32)/\mathbf{Z}_2$.

The Narain moduli space is characterized, up to T duality equivalences that will be discussed below, by a symmetric matrix $M'_{ab} \in O(n, 16 + n)$, which satisfies $M' \eta M' = \eta$. The fact that it is symmetric means that it actually parametrizes the coset space $O(n, 16 + n)/O(n) \times O(16 + n)$, which has $n(16 + n)$ real dimensions. To describe the T duality equivalences, it is convenient to change to the basis defined by the basis vectors of the self-dual lattice. Accordingly, we define

$$M_{IJ} = V_I^a M'_{ab} V_J^b = (V^T M' V)_{IJ}. \quad (4)$$

This matrix is also symmetric and satisfies

$$M L^{-1} M = L, \quad (5)$$

from which it follows that $(L^{-1}M)^2 = 1$. This allows us to define projection operators

$$\mathcal{P}_{\pm} = \frac{1}{2}(1 \pm L^{-1}M). \quad (6)$$

\mathcal{P}_+ projects onto an n -dimensional subspace, which will correspond to right-movers. Similarly, \mathcal{P}_- projects onto the $(16+n)$ -dimensional space of left-movers.

The theory we are seeking should be invariant under an infinite discrete group of T duality transformations, denoted $\Gamma_{n,16+n}$,⁴ so that the actual moduli space is the standard Narain space

$$\mathcal{M}_{n,16+n} = \Gamma_{n,16+n} \backslash O(n, 16+n) / O(n) \times O(16+n). \quad (7)$$

The lagrangian will be formally invariant under an $O(n, 16+n)$ transformation. In terms of coordinates $Y' = VY$, the transformation is just $M' \rightarrow \Lambda^T M' \Lambda$ and $Y' \rightarrow \Lambda Y'$, where $\Lambda \in O(n, 16+n)$. In terms of unprimed quantities, this is $Y \rightarrow \tilde{\Lambda} Y$ and $M \rightarrow \tilde{\Lambda}^T M \tilde{\Lambda}$, where $\tilde{\Lambda} = V^{-1} \Lambda V$. This is actually only a true symmetry of the theory when $Y \rightarrow \tilde{\Lambda} Y$ respects the periodicities of the Y 's. In other words, the T duality transformations form a discrete group $\Gamma_{n,16+n}$ for which all the entries of the matrix $\tilde{\Lambda}$ are integers.

The desired equations of motion for the Y coordinates are [12, 13, 14, 15]

$$\mathcal{P}_- \partial_+ Y = 0 \quad \text{and} \quad \mathcal{P}_+ \partial_- Y = 0, \quad (8)$$

where $\xi^{\pm} = \xi^1 \pm \xi^0$, so that $\partial_{\pm} = \frac{1}{2}(\partial_1 \pm \partial_0)$. ξ^0 and ξ^1 are the world-sheet time and space, respectively. The pair of equations in (8) can be combined in the form

$$M \partial_0 Y - L \partial_1 Y = 0. \quad (9)$$

It is easy to write down a lagrangian that gives this equation:

$$\mathcal{L}_N = \frac{1}{2}(\partial_0 Y M \partial_0 Y - \partial_0 Y L \partial_1 Y). \quad (10)$$

This is just (a modest generalization of) the lagrangian for a chiral 2d boson first proposed by Floreanini and Jackiw [16]. Two things are peculiar about this lagrangian. First, it does not have manifest Lorentz invariance. However, in ref. [17] it was shown that \mathcal{L}_N has a global symmetry that can be interpreted as describing a non-manifest Lorentz invariance. Second, it gives the equation of motion

$$\partial_0[M \partial_0 Y - L \partial_1 Y] = 0, \quad (11)$$

⁴It is often called $O(n, 16+n; \mathbf{Z})$.

which has a second, unwanted, solution $Y^I = f^I(\xi^1)$. The resolution of the second problem is quite simple. The transformation $\delta Y^I = f^I(\xi^1)$ is a gauge symmetry of \mathcal{L}_N , and therefore $f^I(\xi^1)$ represents unphysical gauge degrees of freedom.

The first problem, the noncovariance of \mathcal{L}_N , is more interesting. This paper will follow the PST approach [8], and extend \mathcal{L}_N to a manifestly Lorentz invariant action by introducing an auxiliary scalar field $a(\xi)$. The desired generalization of \mathcal{L}_N is then

$$\mathcal{L}_{PST} = \frac{1}{2(\partial a)^2} (\tilde{Y} M \tilde{Y} - \tilde{Y} L \partial Y \cdot \partial a), \quad (12)$$

where

$$\tilde{Y}^I = \epsilon^{\alpha\beta} \partial_\alpha Y^I \partial_\beta a. \quad (13)$$

Also, $(\partial a)^2$ and $\partial Y \cdot \partial a$ are formed using the 2d Lorentz metric, which is diagonal with $\eta^{00} = -1$ and $\eta^{11} = 1$.

The theory given by \mathcal{L}_{PST} has two gauge invariances. The first is

$$\begin{aligned} \delta Y &= \varphi \left(\frac{1}{\partial_+ a} \mathcal{P}_- \partial_+ Y + \frac{1}{\partial_- a} \mathcal{P}_+ \partial_- Y \right), \\ \delta a &= \varphi, \end{aligned} \quad (14)$$

where $\varphi(\xi^0, \xi^1)$ is an arbitrary infinitesimal scalar function. If this gauge freedom is used to set $a = \xi^1$, then \mathcal{L}_{PST} reduces to \mathcal{L}_N . The second gauge invariance is

$$\delta Y^I = f^I(a), \quad \delta a = 0, \quad (15)$$

where f^I are arbitrary infinitesimal functions of one variable. This is the covariant version of the gauge symmetry of \mathcal{L}_N that was used to argue that the undesired solution of the equations of motion is pure gauge.

2.2 Reparametrization Invariant Action

The formulas described above are not the whole story of the bosonic degrees of freedom of the toroidally compactified heterotic string, because they lack the Virasoro constraint conditions. The standard way to remedy this situation is to include an auxiliary world-sheet metric field $g_{\alpha\beta}(\xi)$, so that the world-sheet Lorentz invariance is replaced by world-sheet general coordinate invariance. Since we now want to include the coordinates X^m describing the uncompactified dimensions, as well, let us also introduce an induced world-sheet metric

$$G_{\alpha\beta} = g_{mn}(X) \partial_\alpha X^m \partial_\beta X^n, \quad (16)$$

where $g_{mn}(X)$ is the string frame target-space metric in d dimensions. It is related to the canonically normalized metric by a factor of the form $\exp(\alpha\phi)$, where ϕ is the dilaton and α is a numerical constant, which can be computed by requiring that the target-space lagrangian is proportional $\exp(-2\phi)$. We will mostly be interested in taking ϕ to be a constant and g_{mn} to be proportional to the flat Minkowski metric. Then the heterotic string coupling constant is $\lambda_H = \exp \phi$, and the desired world sheet lagrangian is

$$\mathcal{L}_g = -\frac{1}{2}\sqrt{-g}g^{\alpha\beta}G_{\alpha\beta} + \frac{\tilde{Y}M\tilde{Y}}{2\sqrt{-g}(\partial a)^2} - \frac{\tilde{Y}L\partial Y \cdot \partial a}{2(\partial a)^2}. \quad (17)$$

Now, of course, $(\partial a)^2 = g^{\alpha\beta}\partial_\alpha a\partial_\beta a$ and $\partial Y \cdot \partial a = g^{\alpha\beta}\partial_\alpha Y\partial_\beta a$. The placement of the $\sqrt{-g}$ factors reflects the fact that $\tilde{Y}/\sqrt{-g}$ transforms as a scalar. The complete lagrangian should also contain a term of the form $\phi\sqrt{-g}R$, where R is the world-sheet curvature scalar. However, this term is an order α' correction, which is beyond the scope of the present analysis.

There are a few points to be made about \mathcal{L}_g . First of all, the PST gauge symmetries continue to hold, so it describes the correct degrees of freedom. Second, just as for more conventional string actions, it has Weyl invariance: $g_{\alpha\beta} \rightarrow \lambda g_{\alpha\beta}$ is a local symmetry. This ensures that the stress tensor

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta S_g}{\delta g^{\alpha\beta}}, \quad (18)$$

is traceless ($g^{\alpha\beta}T_{\alpha\beta} = 0$).

Using the general coordinate invariance to choose $g_{\alpha\beta}$ conformally flat, and using the PST gauge invariance to set $a = \xi^1$, the Y equations of motion reduce to those described in the previous subsection. In addition, one obtains the classical Virasoro constraints $T_{++} = T_{--} = 0$. With these gauge choices one finds

$$T_{++} = G_{++} + \partial_0 Y(M\partial_0 Y - L\partial_1 Y) + \partial_0 Y L\partial_+ Y. \quad (19)$$

Imposing the equation of motion $M\partial_0 Y - L\partial_1 Y = 0$, this can be recast in the form

$$T_{++} = G_{++} + \partial_+ Y L\partial_+ Y, \quad (20)$$

which is the standard result for the left-moving stress tensor. In similar fashion one finds

$$T_{--} = G_{--} + \partial_- Y L\partial_- Y, \quad (21)$$

for the right-movers.

2.3 Elimination of the Auxiliary Metric

The lagrangian \mathcal{L}_g is written with an auxiliary world-volume metric, which is called the Howe–Tucker or Polyakov formulation. This is the most convenient description for many purposes. However, for the purpose of comparing to expressions derived from the M5-brane later in this paper, it will be useful to also know the version of the lagrangian in which the auxiliary metric is eliminated — the Nambu–Goto formulation.

Note that \mathcal{L}_g only involves the metric components in the combination $\sqrt{-g}g^{\alpha\beta}$. Let us therefore define

$$p = \sqrt{-g}g^{11} \quad \text{and} \quad q = \sqrt{-g}g^{10}. \quad (22)$$

Since $\det(\sqrt{-g}g^{\alpha\beta}) = -1$, the remaining component is

$$\sqrt{-g}g^{00} = \frac{q^2 - 1}{p}. \quad (23)$$

With these definitions, and setting $a = \xi^1$ for simplicity, \mathcal{L}_g in eq. (17) takes the form

$$\mathcal{L}_g = pA_1 + qA_2 + \frac{q^2}{p}A_3 + \frac{1}{p}A_4 + \frac{q}{p}A_5 + A_6, \quad (24)$$

where

$$\begin{aligned} A_1 &= -\frac{1}{2}G_{11}, & A_2 &= -G_{01}, & A_3 &= -\frac{1}{2}G_{00} \\ A_4 &= \frac{1}{2}(G_{00} + \tilde{Y}M\tilde{Y}) \\ A_5 &= -\frac{1}{2}\tilde{Y}L\partial_0 Y, & A_6 &= -\frac{1}{2}\tilde{Y}L\partial_1 Y. \end{aligned} \quad (25)$$

It is straightforward to form and solve the p and q equations of motion. Substituting the solutions back in \mathcal{L}_g gives the lagrangian

$$\mathcal{L} = -\sqrt{4\left(A_1 - \frac{A_2^2}{4A_3}\right)\left(A_4 - \frac{A_5^2}{4A_3}\right) + A_6 - \frac{A_2A_5}{2A_3}}. \quad (26)$$

Substituting the expressions for the A 's, given above, and reinstating the a dependence, leaves the final form for the bosonic part of the heterotic string in $10 - n$ dimensions

$$\mathcal{L} = -\sqrt{-G}\sqrt{1 + \frac{\tilde{Y}M\tilde{Y}}{G(\partial a)^2} + \left(\frac{\tilde{Y}L\tilde{Y}}{2G(\partial a)^2}\right)^2} - \frac{\tilde{Y}L\partial Y \cdot \partial a}{2(\partial a)^2}, \quad (27)$$

where $G = \det G_{\alpha\beta}$, and now

$$(\partial a)^2 = G^{\alpha\beta}\partial_\alpha a\partial_\beta a, \quad \partial Y \cdot \partial a = G^{\alpha\beta}\partial_\alpha Y\partial_\beta a. \quad (28)$$

3 Wrapping the M-Theory Five-Brane on K3

Let us now consider double dimensional reduction of the M5-brane on K3.⁵ Our starting point is the bosonic part of the M5-brane action [4] in the general coordinate invariant PST formulation of [5]⁶

$$\mathcal{L}_5 = -\sqrt{-\det\left(G_{\mu\nu} + i\frac{G_{\mu\rho}G_{\nu\sigma}\tilde{H}^{\rho\sigma}}{\sqrt{-(\det G_{\mu\nu})(\partial a)^2}}\right)} - \frac{\tilde{H}^{\mu\nu}H_{\mu\nu\rho}G^{\rho\lambda}\partial_\lambda a}{4(\partial a)^2}, \quad (29)$$

where $G_{\mu\nu} = g_{MN}(X)\partial_\mu X^M\partial_\nu X^N$ is the induced world-volume metric, $H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}$, $(\partial a)^2 = G^{\mu\nu}\partial_\mu a\partial_\nu a$, and

$$\tilde{H}^{\mu\nu} = \frac{1}{6}\epsilon^{\mu\nu\rho\sigma\tau\lambda}H_{\rho\sigma\tau}\partial_\lambda a. \quad (30)$$

Refs. [4, 7] only considered flat backgrounds. However, the generalization to include $g_{MN}(X)$ in the way indicated is certainly allowed, provided that it solves the 11d supergravity field equations. Since the other 11d fields are still assumed to vanish, $g_{MN}(X)$ must be Ricci flat. We will take it to be a product of a Ricci-flat K3 and a flat 7d Minkowski space-time.

The action $\int \mathcal{L}_5 d^6\sigma$ has several types of local symmetries. The two manifest ones are 6d general coordinate invariance and invariance under a gauge transformation $\delta B = d\Lambda$. In addition, there are two PST gauge invariances [5]. One, with parameters $\phi_\mu(\sigma)$, is given by⁷

$$\delta B_{\mu\nu} = \phi_\mu\partial_\nu a - \phi_\nu\partial_\mu a, \quad (31)$$

while a and X^M remain invariant. The other, with parameter $\varphi(\sigma)$, is given by $\delta a = \varphi$, $\delta X^M = 0$, and

$$\delta B_{\mu\nu} = \frac{1}{(\partial a)^2}\varphi H_{\mu\nu\rho}G^{\rho\lambda}\partial_\lambda a - 2\varphi\frac{\partial L_1}{\partial \tilde{H}^{\mu\nu}}, \quad (32)$$

where L_1 is the first term in eq. (29).

Since the M5-brane is taken to wrap the spatial K3, the diffeomorphism invariances of the M5-brane action in these dimensions can be used to equate the four world-volume coordinates that describe the K3 with the four target-space coordinates that describe the K3. In other words, we set $\sigma^\mu = (\xi^\alpha, x^i)$ and $X^M = (X^m, x^i)$. Note that Latin indices i, j, k

⁵See ref. [18] for a review of the mathematics of K3 and some of its appearances in string theory dualities.

⁶The supersymmetric extension including the fermionic θ variables is described in refs. [6, 7].

⁷Actually, this symmetry and $\delta B = d\Lambda$ can be combined and generalized to $\delta B = f(a)d\Lambda$, where f is an arbitrary function of a and Λ is an arbitrary one-form. This is the two-form counterpart of eq. (15).

are used for the K3 dimensions (x^i) and early Greek letters for the directions (ξ^α), which are the world-sheet coordinates of the resulting string action. This wrapping by identification of coordinates, together with the extraction of the K3 zero modes, is what is meant by double dimensional reduction. With these choices, the 6d metric can be decomposed into blocks

$$(G_{\mu\nu}) = \begin{pmatrix} \tilde{G}_{\alpha\beta} & 0 \\ 0 & h_{ij} \end{pmatrix}, \quad (33)$$

with h_{ij} and $\tilde{G}_{\alpha\beta}$ being the K3 metric and the induced metric on the string world-sheet, respectively. The purpose of the tilde is to emphasize that $\tilde{G}_{\alpha\beta} = \tilde{g}_{mn} \partial_\alpha X^m \partial_\beta X^n$, where \tilde{g}_{mn} is the 7d part of the canonical 11d metric. It differs from g_{mn} of sect. 2 by a scale factor, which will be determined below. It is convenient to take the PST scalar field a to depend on the ξ^α coordinates only. This amounts to partially fixing a gauge choice for the PST gauge invariance.

The two-form field B has the following contributions from K3 zero modes:

$$B_{ij} = \sum_{I=1}^{22} Y^I(\xi) b_{Iij}(x), \quad B_{\alpha i} = 0, \quad B_{\alpha\beta} = c_{\alpha\beta}(\xi), \quad (34)$$

where b_{Iij} are the 22 harmonic representatives of $H^2(\text{K3}, \mathbf{Z})$, the integral second cohomology classes of K3. Any other terms are either massive or can be removed by gauge transformations. The nonzero components of $H_{\mu\nu\rho}$ and $\tilde{H}^{\mu\nu}$ are

$$H_{\alpha ij} = \sum_{I=1}^{22} \partial_\alpha Y^I b_{Iij} \quad (35)$$

$$\tilde{H}^{ij} = \sum_{I=1}^{22} \tilde{Y}^I \frac{1}{2} \epsilon^{ijkl} b_{Ikl} = \sum_{I=1}^{22} \sqrt{h} \tilde{Y}^I (*b_I)^{ij}, \quad (36)$$

where $\tilde{Y}^I = \epsilon^{\alpha\beta} \partial_\alpha Y^I \partial_\beta a$ as in eq. (13). Note that $c_{\alpha\beta}$ does not contribute.

We need to understand why the coefficients Y^I should be periodically identified with period 2π , since they will be identified with the Y^I coordinates of the heterotic string action in sect. 2. To show why that is so, consider a transformation $B \rightarrow B + b^{(2)}$ with $db^{(2)} = 0$. This leaves invariant the lagrangian (29), which only depends on $H = dB$. However, in the case of a general background, the action also contains the term $-\frac{1}{2\pi} \int_{\mathcal{Y}} B \wedge G_4$, which could fail to be invariant. Here G_4 is the closed four-form background field strength of 11d supergravity. It must satisfy the topological restriction [19]

$$\left[\frac{G_4}{2\pi} \right] - \frac{p_1(\mathcal{Y})}{4} \in H^4(\mathcal{Y}, \mathbf{Z}), \quad (37)$$

where $p_1(\mathcal{Y})$ is the first Pontryagin class of the M5-brane world-volume manifold \mathcal{Y} . In the present case, $\mathcal{Y} = K3 \times \Sigma$, where Σ is identified as the 2d world-sheet of the resulting heterotic string. The value of the first Pontryagin class is divisible by 4 on all closed four-cycles of this manifold, and therefore the p_1 term can be dropped. It remains to determine the restrictions on $b^{(2)}$ that arise from requiring that the change of the M5-brane action $\delta S = -\frac{1}{2\pi} \int b^{(2)} \wedge G_4$ is an integral multiple of 2π . $b^{(2)}$ can be written as a superposition of two-forms Poincaré dual to the four-cycles of the 6d world volume. In the case at hand, the possible four-cycles are K3 itself and Σ times the 2-cycles of the K3. The dual 2-forms are ω_2 , which is the normalized volume form of the world sheet Σ , and b_I , the 22 two-forms of K3 introduced earlier. So, writing

$$\frac{1}{2\pi} b^{(2)} = n^0 \omega_2 + \sum_{I=1}^{22} n^I b_I, \quad (38)$$

where n^0 and n^I are integers, it follows that

$$\delta S = - \left(n^0 \int_{K3} G^{(4)} + \sum_{I=1}^{22} n^I \int_{C_I \times \Sigma} G^{(4)} \right) \quad (39)$$

is an integer multiple of 2π . Thus, $B \rightarrow B + 2\pi(n^0 \omega_2 + \sum_{I=1}^{22} n^I b_I)$, which corresponds to shifting Y^I by $2\pi n^I$, can be interpreted as a “large gauge transformation.” This implies that the Y^I are coordinates of the Jacobian $H^2(K3, \mathbf{R})/H^2(K3, \mathbf{Z})$. The n^0 term describes a shift of the coordinate $c_{\alpha\beta}$ in eq. (34). As noted earlier, it drops out of the formulas.

Now we can compute the string action that arises from double dimensional reduction. Substituting the decompositions (35) and (36) into eq. (29), gives for the second term in the action

$$\begin{aligned} -\frac{\tilde{H}^{\mu\nu} H_{\mu\nu\rho} G^{\rho\lambda} \partial_\lambda a}{4(\partial a)^2} &= -\frac{\tilde{Y}^I (\epsilon^{ijkl} b_{Iij} b_{Jkl}) \partial_\alpha Y^J \tilde{G}^{\alpha\beta} \partial_\beta a}{8(\partial a)^2} \\ &= -\frac{\tilde{Y}^I \sqrt{h} * (b_I \wedge * b_J) \partial_\alpha Y^J \tilde{G}^{\alpha\beta} \partial_\beta a}{2(\partial a)^2}. \end{aligned} \quad (40)$$

Here $(\partial a)^2 = \tilde{G}^{\alpha\beta} \partial_\alpha a \partial_\beta a$. Substituting eq. (33), the determinant in the first term of eq. (29) becomes

$$\det \left(G_{\mu\nu} + i \frac{G_{\mu\rho} G_{\nu\sigma} \tilde{H}^{\rho\sigma}}{\sqrt{-(\det G_{\mu\nu})(\partial a)^2}} \right) = \tilde{G} h \left(1 + \frac{1}{2} \text{tr}(\tilde{A}^2) + \det(\tilde{A}) \right), \quad (41)$$

where $\tilde{G} = \det \tilde{G}_{\alpha\beta}$, $h = \det h_{ij}$, and

$$\tilde{A}_i^j = \frac{1}{\sqrt{(-\tilde{G})h(\partial a)^2}} h_{ik} \tilde{H}^{jk}. \quad (42)$$

Substituting eq. (34) gives

$$\frac{1}{2}\text{tr}(\tilde{A}^2) = \frac{\tilde{Y}^I b_{IJ} b_J^{ij} \tilde{Y}^J}{2\tilde{G}(\partial a)^2} = \frac{\tilde{Y}^I * (b_I \wedge *b_J) \tilde{Y}^J}{\tilde{G}(\partial a)^2} \quad (43)$$

and

$$\begin{aligned} \det(\tilde{A}) &= \det\left(\frac{h_{ij} \tilde{Y}^I (*b_I)^{jk}}{\sqrt{-\tilde{G}(\partial a)^2}}\right) = \frac{1}{(-\tilde{G}(\partial a)^2)^2} \frac{1}{h} \left(\text{Pf}\left(\tilde{Y}^I (*b_I)_{ij}\right)\right)^2 \\ &= \frac{1}{4} \frac{\left(\tilde{Y}^I * (*b_I \wedge *b_J) \tilde{Y}^J\right)^2}{(-\tilde{G}(\partial a)^2)^2}. \end{aligned} \quad (44)$$

To make the connection with the heterotic string action of the previous section, we make the identifications

$$L_{IJ} = \int_{K3} b_I \wedge b_J, \quad (45)$$

$$M_{IJ} = \int_{K3} b_I \wedge *b_J. \quad (46)$$

Note that $*b_I = b_J (L^{-1}M)^J{}_I$, and therefore $(L^{-1}M)^2 = 1$, as in sect. 2. Note also that $b_I \wedge b_J$ and $b_I \wedge *b_J$ are closed four-forms, and therefore they are cohomologous to the unique harmonic four-form of the K3, which is the volume form ω . It follows that⁸

$$b_I \wedge b_J = *b_I \wedge *b_J = \frac{L_{IJ}}{\mathcal{V}} \omega + dT_{IJ}, \quad b_I \wedge *b_J = \frac{M_{IJ}}{\mathcal{V}} \omega + dU_{IJ}, \quad (47)$$

where $\mathcal{V} = \int_{K3} \omega$ is the volume of the K3 and $U_{IJ} = T_{IK} (L^{-1}M)^K{}_J$. The exact terms are absent when either two-form is self-dual, but there is no apparent reason why they should vanish when both of them are anti-self-dual. If we nevertheless ignore the exact pieces in these formulas, substitute into eq. (29) using eqs (41) – (44), and integrate over the K3, we obtain

$$\mathcal{L}_1 = -\mathcal{V} \sqrt{-\tilde{G}} \sqrt{1 + \frac{\tilde{Y}^I M_{IJ} \tilde{Y}^J}{\tilde{G}(\partial a)^2 \mathcal{V}} + \frac{1}{4} \left(\frac{\tilde{Y}^I L_{IJ} \tilde{Y}^J}{\tilde{G}(\partial a)^2 \mathcal{V}} \right)^2} - \frac{\tilde{Y}^I L_{IJ} \partial_\alpha Y^J \tilde{G}^{\alpha\beta} \partial_\beta a}{2(\partial a)^2}. \quad (48)$$

This is precisely the heterotic string lagrangian (for $n = 3$) presented in eq. (27) of the previous section provided that the 7d metric g_{mn} in the string frame is related to the metric \tilde{g}_{mn} derived from 11d by

$$g_{mn} = \mathcal{V} \tilde{g}_{mn} \quad (49)$$

⁸In the first version of this paper the exact terms were omitted from these formulas. We are grateful to G. Moore for pointing out the error to us.

so that $G_{\alpha\beta} = \mathcal{V}\tilde{G}_{\alpha\beta}$. This is the same scaling rule found by a different argument in ref. [1]. Then, following ref. [1], the Einstein term in the 7d lagrangian is proportional to $\mathcal{V}\sqrt{-\tilde{g}}R(\tilde{g}) = \mathcal{V}^{-3/2}\sqrt{-g}R(g)$, from which one infers that $\mathcal{V} \sim \lambda_H^{4/3}$.

To complete the argument we must still explain why terms that have been dropped make negligible contributions. It is not at all obvious that the exact pieces in eq. (47) can be neglected. However, this is what is required to obtain the desired answer, so we are confident that it must be true. We hope to return to this point in the future. The other class of terms that have been dropped are the Kaluza–Klein excitations of the five-brane on the K3. By simple dimensional analysis, one can show that in the heterotic string metric these contributions to the mass-squared of excitations are of order λ_H^{-2} . Therefore they represent non-perturbative corrections from the heterotic viewpoint. Since our purpose is only to reproduce the perturbative heterotic theory, they can be dropped. Another class of contributions, which should not be dropped, correspond to simultaneously wrapping the M2-brane around a 2-cycle of the K3. These wrappings introduce charges for the 22 U(1)’s, according to how many times each cycle is wrapped. The contribution to the mass-squared of excitations depends on the shape of the K3, of course, but in the heterotic metric it is independent of its volume and hence of the heterotic string coupling constant.

We are grateful to Michael Douglas and David Lowe for reading the manuscript. We also wish to thank Greg Moore and Malcolm Perry for helpful discussions.

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